On the number of points in a lattice polytope

Arseniy Akopyan* Makoto Tagami†

Abstract

In this article we will show that for every natural d and $n > 1$ there exists a natural number t such that every d-dimensional simplicial complex $\mathcal{T}$ with vertices in $\mathbb{Z}^d$ scaled in t times contains exactly $\chi(\mathcal{T})$ modulo n lattice points, where $\chi(\mathcal{T})$ is the Euler characteristic of $\mathcal{T}$.

This problem given to one of the authors by Rom Pinchasi. He noticed that if you scale a segment with vertices in a lattice in two times then the number of lattice points in the scaled segment will be odd. For polygons with vertices in a two-dimensional lattice, the same fact follows from Pick's formula and this polygon must be scaled in four times. We will show that the following theorem holds:

Theorem 1. *For any natural numbers d and $n > 1$ there exists a natural number t such that if $\mathcal{T}$ is any simplicial complex in $\mathbb{R}^d$ with vertices in the integer lattice $\mathbb{Z}^d$ then the number of lattice points in the polytope $t\mathcal{T}$ is equivalent to $\chi(\mathcal{T})$ modulo n.*

Here $\chi(\mathcal{T})$ is the Euler characteristic of the complex $\mathcal{T}$ and $t\mathcal{T}$ denote the image of $\mathcal{T}$ under similarity with the center at the origin and ratio equal to t.

First we prove the following lemma.

Lemma 2. *Let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^d$ with vertices in the integer lattice $\mathbb{Z}^d$, p be any prime number and $l = \left[\log_p d\right]$. Then for any natural $k > l$ the convex polytope $p^k\mathcal{P}$ contains exactly one modulo p^{k-l} point from the lattice $\mathbb{Z}^d$.*

Proof. From Stanley's nonnegativity theorem (more precisely Lemma 3.14 in [1]) it follows that in this case the number of lattice points in the convex polytope $t\mathcal{P}$ equals exactly:

$$\binom{t+d}{d} + h_1\binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d\binom{t}{d},$$

where $h_1, h_2, \ldots, h_d$ are nonnegative integer numbers.

*University of Texas at Brownsville (USA), Institute of Systems Analysis (Russia). Research supported by RFBR grants 08-01-00565-a and 10-01-00096-a

†Mathematical Institute, Tohoku University

Suppose $t = p^k$ and $m \le d \le p^{l+1} - 1$. If α is the maximal power of p which divides m then $(m + p^k)/p^\alpha \equiv m/p^\alpha \pmod{p^{k-l}}$. Using this fact it is easy to show that $\binom{t+d}{d} \equiv 1 \pmod{p^{k-l}}$. Also from Kummer's theorem it follows that for any $i = 1, 2, \dots, d$ we have $\binom{t+d-i}{d} \equiv 0 \pmod{p^{k-l}}$. So as we can see, the number of lattice points equals exactly one modulo p^{k-l}. □

Now it is easy to prove the general case.

Proof of Theorem 1. Consider the prime factorization of n:

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_s^{\alpha_s}.$$

Suppose $\beta_i = \alpha_i + \left[\log_{p_i} d\right]$. Define $t = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \dots p_s^{\beta_s}$.

Suppose Δ is a simplex. By Lemma 2 we have that the number of lattice points in $t\Delta$ equals 1 modulo $p_i^{\alpha_i}$ for any $i = 1, 2, \dots, s$. From the Chinese remainder theorem, it follows that this number is equivalent to 1 modulo n.

We know that the Euler characteristic of every simplex (with its interior) equals 1 and the Euler characteristic is an additive function on simplicial complexes. Since the number of lattice points modulo n is also an additive function, we obtain that the number of lattice points is equivalent to exactly $\chi(\mathcal{T}) \pmod{n}$.

□

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References

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